each side (91 for the Evens, and 190 for the Odds) is replaced by a pyramid. These pyramids are formed by superposing squares (thin prisms), and are used to call attention to the peculiar construction of the numbers which they represent. Thus $91=6^2+5^2+4^2+3^2+2^2+1$, and $190=8^2+7^2+6^2+5^2+4^2$. Since the former contained the squares of all numbers from 1 to 6, it was called a perfect pyramid, but since the latter lacked $3^2+2^2+1^2$ it was known as *tricurta* (thrice curtailed),

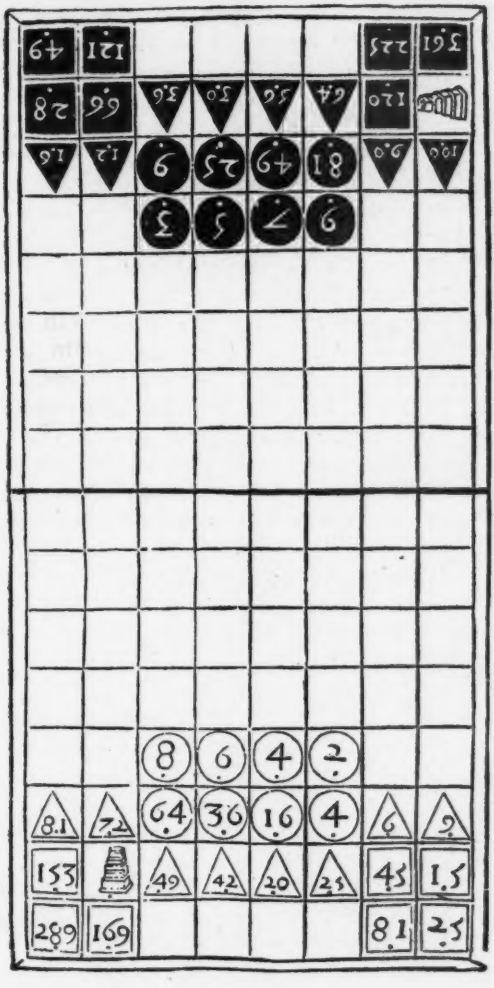

Fig. 1. From Boissière's work of 1554, 1556.

The lower row of squares is obtained from the upper one by a formula somewhat like the one used in obtaining the lower row of triangles. In the case of the squares, if we call the number in the circle at the top n, and the number in the upper square s, then the number in the lower square is $\left(\frac{2n+1}{n+1}\right)s$. Thus $\left(\frac{2 \cdot 2+1}{2+1}\right) \cdot 15 = \frac{5}{3} \cdot 15 = 25$; $\left(\frac{2 \cdot 4+1}{4+1}\right) \cdot 45 = \frac{9}{5} \cdot 45 = 81$, and so on. The ratio between 25 and 15, $\frac{5}{3}$, is one of the superpartientes, namely, the superbipartientes (surpassing by two parts). It should be said, however, that other rules are given for the derivation of these numbers, and that there are slight variations in the pieces, but these have no significance.

The pieces are now arranged for the opening of the play in the manner shown in Fig. 1. The players move the pieces in turn, as in chess. A circle moves one space, a triangle three, and a square four. The game consists in capturing an opponent's pieces, this being effected in one of four ways—by meeting, by assault, by ambuscade, and by siege.

The method by meeting may be illustrated as follows: If Even's triangle 25 can, by advancing three spaces, reach Odd's circle 25, Even does not move his piece, but simply takes up his opponent's.

The capture by assault is affected in this way: If a smaller number, multiplied by the number of vacant spaces between it and a larger one equals the larger one, it may take it. For example, Odd's circle 5 may take Even's square 45 if nine separates the two. This requires the players to be familiar with the multiplication table, and for this purpose Fortolfus provides the usual square array known in the Middle Ages as the *mensa Pythagorica.*

The capture by ambuscade is as follows: If two pieces whose sum equals the number on an opponent's piece can be moved into the spaces on either side of it, the latter is ambuscaded and must surrender. For example, to capture Odd's triangle 12, Even's circles 4 and 8 must be able to move on either side of it.

The capture by siege is effected if a piece is immediately surrounded on all four sides by opposing pieces; that is, if the adjacent spaces above, below, to the right, and to the left are filled.

It is evident that a pyramid can rarely be taken except by siege. Interest was therefore added to the game by making it subject to several attacks. A pyramid was considered to be in danger whenever one of its laminæ was attacked by any one of the four methods. In this case a ransom was allowable, namely, a piece of the same value as the base. In case no such piece could be offered because of prior capture, any other piece could be given that the opponent might be willing to accept. Positive capture of the piece not being possible if the numbers 91 and 190 were retained, it was permitted if the base square was successfully attacked, namely, 36 or 64. The piece having no particular value, its loss was no more serious than that of any other piece, but the plans of attack were more interesting.

Fig. 2. The Setting of the Pieces.

As already stated, the game consists in capturing an opponent's pieces. This, however, is not all there is of it. The capture is undertaken for the purpose of obtaining what is technically called a Victory, and the rules provide for no less than eight of these Victories. Before beginning to play, the particular kind of Victory for which the contest is to be waged is agreed upon by the parties. Five of these kinds are known as Common Victories, and the rest as Proper Victories, the former being considered as suited to tyros and the latter as worthy of veteran players.

Common Victories were, as already said, of five kinds, as follows: (1) Victory *de corpore*, decided by the number of pieces captured; (2) Victory *de bonis*, depending upon the value of the pieces; (3) Victory *de lite*, depending not only upon the value of the pieces but upon the number of the digits inscribed upon them; (4) Victory *de honore*, depending upon both the number of the pieces and their value; (5) Victory *de honore liteque*, depending upon the number of pieces, their value, and the number of digits inscribed upon them.

If, for example, the players decided upon the Victory *de corpore*, they would agree in advance upon some number, as twenty-four, as the winning number. As soon as either player captured twenty-four of the opponent's pieces he won the game.

If they decided in advance upon the Victory *de bonis*, they would agree upon some number like 160 as the winning number. Each player would then seek to capture pieces of which the sum of the values should equal or exceed 160.

If they decided upon the Victory *de lite* they might again select 160 with the further condition that the total number of digits on the pieces should equal some small number such as eight. A player would then try to capture pieces like 56, 64, 28, and 15, but would not try for 121, 9, and 30.

If the victory was to be *de honore* the players would agree upon some number like 160 for the sum of the values, and some other number like five for the number of the pieces. In this case neither 56, 64, 28, 12 nor 121, 9, 30 would suffice, but 64, 36, 30, 25, 5 would meet the two conditions.

In the Victory *de honore liteque* the players might agree upon 160 for the values, five for the number of pieces, and nine for the number of digits. These conditions are satisfied by 64, 36, 30, 25, 5 from the Odd's pieces, and 64, 36, 42, 16, 2 from the Even's pieces.

It is already evident that the game has more of merit than at first seemed probable. These Common Victories do not, however, show it as played by the real lover of Rithmomachia, It was in the Proper Victories that he found a game worthy of his efforts, and with these we shall close this description.

The Proper Victories were known by the names of Magna, Major, and Praestantissima, and they resulted from combinations relating to the three best known types of progressions, the arithmetic, geometric, and harmonic—

progressions that had come down through the Greek mathematics from the Pythagoreans. In each of these victories the pieces, one of which must be taken from the opposing side, must be displayed in the selected progression from the opponent's side of the board.

The Victoria Magna consists in arranging three counters in any one of the three simple progressions. There are forty-one combinations that make possible such an arrangement in arithmetic progression, eighteen in geometric progression, and seventeen in harmonic progression. The possibilities are greater for Even in the first case and for Odd in the second case, and they are equal in the third case. One of these arrangements, in geometric progression, is 6, 8, 12. To this Fortolfus gave the name Cubic Victory, the first number representing the faces, the second the vertices, and the third the edges of a cube.

The earlier writers gave to the second of the Proper Victories the name Victoria Minor, but Boissière calls it Victoria Major. In this there are combined two progressions, arithmetic and geometric, geometric and harmonic, or harmonic and arithmetic. To secure this victory four pieces must be brought in line in the enemy's field, two of which must belong to one of the selected progressions and two to the other. For example, 2, 3, 4, 8 would gain a Victoria Major for either Even or Odd, for 2, 3, 4 are in arithmetic progression and 2, 4, 8 are in geometric, where 2, 4, 8 are Even's pieces and 3 is Odd's piece. There are in all sixty-one such double progressions, all but one of which can be used by Even, and all of which can be used by Odd.

The climax of the game was reached in the Victoria Praestantissima, or Victoria Excellentissima. In this victory it was necessary to get four numbers in a row, which numbers embodied all three progressions. There are only six possible solutions to this problem, namely, (2, 3, 4, 6), (4, 6, 8, 12), (7, 8, 9, 12), (4, 6, 9, 12), (3, 5, 15, 25), (12, 15, 16, 20). Of these only one (4, 6, 9, 12) contains four terms in geometric progression, the others containing proportions. Upon these combinations the early writers dilate with not a little affection. For example, in the set 4, 6, 8, 12, the comparison of 12 and 8, or of 6 and 4, is a sesquialtera ($\frac{3}{2}$), corresponding to the fifth in music; 8 and 4, or 12 and 6, have the ratio 2:1, that of the octave or diapason. The ratio 8:6 gives the diatesseron, while 12:4 gives the interval, a twelfth, including both diapason and diapente. The ratio 8:(6−4) gives the interval of two octaves, or a fifteenth, and all of this was shown in graphic form as follows:

Such is a brief description of the game of which Boissière speaks as "*Noblissimus et antiquissimus ludus Pythagoreus qui Rythmomachia nominatur,*"* a game of which we are told many of the devotees were men of no mean reputation. Such leaders of thought as Gerbert, whom his contemporaries called a wizard but made a Pope; Hermannus, whose infirmity gave him the name of Contractus, by which he is commonly known; Robertus Castrensis, who helped to make the Arab learning known;† Nicolaus Horem, who became Bishop of Lisieux in 1377, and who sought to enrich the intellectual world by his teaching of the ancient theory of numbers; Oronce Finé (Orontius Finaeus), who was professor of mathematics in the (later called) Collége de France in 1532; Jacques le Fèvre d'Estaples (Jacobus Faber Stapulensis), the learned tutor of the son of François I; Thomas Bradwardin, who died in 1349 as Archbishop of Canterbury, and who, from his great learning, was known as "Doctor Profundus;" John Shirwood (Shirewode), who died in 1494 as Bishop of Durham—these are the names of some of those who played the game, and several of them composed tractates setting forth its merits. It cannot be revived, since the interest in the number theory for which it stood has passed away, but even some slight understanding of its nature cannot fail to have interest for any one who takes pleasure in mathematics, in education, or in the evolution of both mathematics and education from the ideals of the Greek philosophy to the ideals of the present day.

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* For the full title in facsimile, see Smith, *Rara Arithmetica*, Boston, 1909, p. 272. with other references in the index.

† Professor Karpinski of the University of Michigan, is now working on one of his translations, the Algebra of Al-Khowarazmi.

IDEALS OF A QUADRATIC NUMBER FIELD IN CANONIC FORM.

By W. B. CARVER, Cornell University, Ithaca, New York.

Sommer* shows that any ideal of the quadratic field $k(\sqrt{m})$ may be reduced to a canonic form (i, i_1+i_2w), i, i_1, and i_2 being rational integers, i_2 a factor of both i and i_1, and

$$w\dagger = \begin{cases} \sqrt{m}, & \text{when } m \text{ is not congruent to } 1 \ (4) \\ \dfrac{1+\sqrt{m}}{2}, & \text{when } m \equiv 1 \ (4). \end{cases}$$

Any integer‡ of the ideal may be expressed in the form

$$li+n(i_1+i_2w),$$

l and n being rational integers.

Using a slightly different notation, let us write the canonic form

$$r(s, t+w)$$

r, s, and t being rational integers, $r \neq 0$, and (to make the form unique)

$$s > t \geqq 0 \tag{1.}$$

$s=1$, $t=0$ would give the canonic form of the rational principal ideal (r), and $r=1$, $s=1$, $t=0$ would give the unit ideal containing every integer of the field.

Consider the case $r=1$, or the ideal $(s, t+w)$. s is the highest common factor of all the rational integers of the ideal,§ and it follows at once that t must satisfy the relation

$$\begin{cases} t^2 \equiv m(s) \\ (2t+1)^2 \equiv m(4s) \end{cases} \tag{2.}$$

Conversely, an ideal $(s, t+w)$ is in canonic form if s and t are rational in-

* *Vorlesungen über Zahlentheorie.* Leipzig, B. G. Teubner, 1907. See pp. 36-44. These are the ideals conceived by Dedikind and Treated by Dirichlet and others.

† Whenever double lines are written with a brace throughout this paper, the upper line will be for the case m not congruent to 1 (4) and the lower one for the case m congruent to 1 (4).

‡ The word "integer," unqualified, will be used in this paper to mean a quadratic integer.

§ Sommer, loc. cit., p. 40.

tegers satisfying conditions (1.) and (2.); for it is readily seen that, under these conditions, any integer of the field, $(a_1+b_1w)s+(a_2+b_2w)(t+w)$, may be expressed in the form $ls+n(t+w)$, l and n being rational integers.

The ideal conjugate to $(s,\ t+w)$, is, in canonic form,

$$\begin{cases} (s,\ s-t+w) \\ (s,\ s-t-1+w). \end{cases}$$

From conditions (1.) and (2.) and well-known theorems in the theory of rational integers, it follows that, for a given prime p not a factor of the discriminant of the field $\binom{d=4m}{d=m}$, there are two canonic ideals $(p,\ t+w)$, or none, according as

$$\begin{cases} \left(\frac{m}{p}\right)=1 \text{ or } -1 \\ \left(\frac{m}{4p}\right)=1 \text{ or } -1 \end{cases}$$

When there are two, they are conjugate. If p is a factor of d, there is always one and only one ideal $(p,\ t+w)$. It is self-conjugate; and $t=0$ except in the two cases,

$$\text{(1.) } m\equiv 1 \text{ (4), when } t=\frac{p-1}{2},$$
$$\text{and (2.) } m\equiv 3 \text{ (4) and } p=2, \text{ when } t=1.$$

For any given rational integer s, there will be ideals of the form $(s,\ t+w)$ if and only if m is a quadratic remainder for each prime factor of s. If there are any such ideals, there will be just 2^r where r is the number of *distinct* prime factors of s which are not factors of d.

MULTIPLICATION.

Consider the product $(p,\ t_1+w)(p,\ t_2+w)$, p being a rational prime not a factor of d. Either these ideals are conjugate, in which case their product is the principal ideal $p(1,\ w)$, or else $t_1=t_2$ and the product is the square $(p,\ t_1+w)^2$. If $(p,\ t_1+w)^2=(s,\ t+w)$, s must be a number of $(p,\ t+w)$* and must therefore be divisible by p. Also, since p^2 is a number of $(s,\ t+w)$, s must be a factor of p^2. Hence s is either p or p^2. If $s=p$, then $(p,\ t_1+w)^2$ is $(p,\ t_1+w)$ or its conjugate $(p,\ p-t_1+w)$.† The

* If one ideal is a factor of another, then every integer of the second is an integer of the first: cf. Sommer, loc. cit., p. 46.

† The argument is given for the case m not congruent to 1 (4). A very similar argument disposes of the case m congruent to 1 (4).

first supposition is trivial, being possible only for $(p,\ t_1+w)=(1,\ w)$. If $(p,\ t_1+w)^2=(p,\ p-t_1+w)$, then $p-t_1+w$ would be an integer of the ideal $(p,\ t_1+w)$. This could only be true if p were a factor of $2t_1$; and since $p\neq 2$ and $p>t_1$, it is impossible. Hence $s=p^2$, and t is uniquely determined by the conditions

$$\begin{cases} t^2\equiv m(p^2) \\ (2t+1)^2\equiv m(4p^2) \end{cases}$$

$$t\equiv t_1\,(p)$$

$$\text{and}\quad 0\leqq t<p^2.$$

The condition $t\equiv t_1\,(p)$ must be true because pt_1+pw must be an integer of the product ideal, and hence there must be rational integers l and n such that

$$pt_1+pw=lp^2+n(t+w)$$

and it is evident that $n=p$ and $l=\dfrac{t_1-t}{p}$.

Similarly, it may be shown that

$$(p,\ t_1+w)^n=(p^n,\ t+w),$$

t being uniquely determined by the conditions

$$\begin{cases} t^2\equiv m(p^n) \\ (2t+1)^2\equiv m(4p^n) \end{cases}$$

$$t\equiv t_1\,(p)$$

$$0\leqq t<p^n.$$

If q is a prime factor of d, then $(q,\ t+w)$ is self-conjugate, $(q,\ t+w)^2=q(1,\ w)$, $(q,\ t+w)^{2n}=q^n(1,\ w)$, and $(q,\ t+w)^{2n+1}=q^n(q,\ t+w)$.

Consider next the product $(s_1,\ t_1+w)(s_2,\ t_2+w)$ where s_1 and s_2 are relative primes. If the product is $(s,\ t+w)$, s must be an integer of both the ideals $(s_1,\ t_1+w)$ and $(s_2,\ t_2+w)$, and hence must be divisible by both s_1 and s_2, and therefore by their product s_1s_2. But s_1s_2 is an integer of the ideal $(s,\ t+w)$, and hence is divisible by s. Therefore $s=s_1s_2$, and t is uniquely determined by the conditions

$$\begin{cases} t^2\equiv m(s_1s_2) \\ (t2+1)^2\equiv m(4s_1s_2) \end{cases}$$

$$t \equiv t_1 (s_1)$$
$$t \equiv t_2 (s_2)$$
$$0 \leqq t < s_1 s_2 .$$

Consider the ideal $(s, t+w)$, and let s_1 be any factor (prime or not) of s. Determine t_1 by the conditions $t_1 \equiv t(s_1)$ and $0 \leqq t < s_1$. Then it may be readily shown that every integer of the ideal $(s, t+w)$ is also an integer of the ideal (s_1, t_1+w), and hence (s_1, t_1+w) is a factor of $(s, t+w)$. We can then find a prime ideal factor of $(s, t+w)$ for every prime factor of s; and if we multiply these prime factors together by the laws shown above, their product will evidently be $(s, t+w)$. It follows then that *the ideal* $(s, t+w)$ *has a prime ideal factor for every prime factor of* s, *and that it has no other factors.*

Consider now the general product of $r_1(s_1, t_1+w)$ and $r_2(s_2, t_2+w)$, and let it be written $r_1 r_2 \{r(s, t+w)\}$. In general, s_1 and s_2 will contain some prime factors which are factors of d, and some which are not; also some prime factors common to both s_1 and s_2, and some appearing only in one of the s's. Let the factors not in d be denoted by p's, and those in d by q's; and let the factors common to s_1 and s_2 be indicated by a bar thus, $\bar{p}$, $\bar{q}$. We may then write

$$s_1 = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots q_1 q_2 \cdots \bar{p}_1^{\beta_1} \bar{p}_2^{\beta_2} \cdots \bar{q}_1 \bar{q}_2 \cdots$$

$$\text{and } s_2 = p_{h+1}^{a_{h+1}} p_{h+2}^{a_{h+2}} \cdots q_{k+1} q_{k+2} \cdots \bar{p}_1^{\gamma_1} \bar{p}_2^{\gamma_2} \cdots \bar{q}_1 \bar{q}_2 \cdots$$

(q's may not occur to powers higher than the first).

Every factor p^a or q not common to both s_1 and s_2 will be a factor of s; and will contribute, for the determination of t, the condition

$$t \equiv t_1 \text{ or } t_2 \ (p \text{ or } q) \tag{3.}$$

Every $\bar{q}$ will be a factor of r, and will not appear as a factor in s.

For each $\bar{p}$ we first determine whether $t_1 \equiv t_2 \ (\bar{p})$ or t_1 not congruent to $t_2 \ (\bar{p})$. Those $\bar{p}$'s which are in the second class we will denote by a double bar $\bar{\bar{p}}$. For $\bar{p}$'s for which the congruence holds, $\bar{p}^{\beta+\gamma}$ will be a factor of s, $\bar{p}$ will not appear in r, and we will have the condition

$$t \equiv t_1 \equiv t_2 \ (\bar{p}) \tag{4.}$$

For a $\bar{\bar{p}}$, let δ be the smaller of the numbers β and γ. Then $\bar{\bar{p}}^{\delta}$ will be a factor of r, $\bar{\bar{p}}^{|\beta-\gamma|}$ will be a factor of s, and we will have

$$t \equiv t_1 \ (\bar{\bar{p}}), \text{ if } \beta > \gamma$$
$$\text{or } t \equiv t_2 \ (\bar{\bar{p}}), \text{ if } \gamma > \beta \tag{5.}$$

The factors thus determined will make up r and s, and t will finally be uniquely determined by the condition

$$\begin{cases} t^2 \equiv m \ (s) \\ (2t+1)^2 \equiv m \ (4s) \end{cases} \tag{6.}$$

together with conditions (3.), (4.), and (5.) above. Hence the product of $r_1(s_1, t_1+w)$ and $r_2(s_2, t_2+w)$ may be written $r_1 r_2 r(s, t+w)$, in which

$$r = \bar{q}_1 \ \bar{q}_2 \ \dots \ \overset{=\beta_1}{p_1} \ \overset{=\beta_2}{p_2} \ \dots$$

$$s = \overset{\alpha_1}{p_1} \ \overset{\alpha_2}{p_2} \ \dots \ q_1 \ q_2 \ \dots \ \overset{\alpha_h+1}{p_{h+1}} \ \overset{\alpha_h+2}{p_{h+2}} \ \dots \ q_{k+1} \ q_{k+2} \ \dots \ \underset{-\beta_1+\gamma_1}{p_1} \ \underset{-\beta_2+\gamma_2}{p_2} \ \dots \ \underset{=|\beta_1-\gamma_1|}{p_1} \ \underset{=|\beta_2-\lambda_2|}{p_2} \ \dots$$

and t is given by the conditions (3.)...(6.)

Examples. Consider the product $(399, 182+\sqrt{7})(378, 175+\sqrt{7})$. Here $d=28$, $s_1=399=19.3.7$, $s_2=378=2.3^3.7$, $s=19.2.3^2$; $t \equiv 175(2)$, $t \equiv 175(3)$, $t \equiv 182(19)$, $t^2 \equiv 7(19.2.3^2)$, and $0 \leqq t < (19.2.3^2)$, which makes $t=103$; $r=7.3$. Hence the product is $21(342, 103+\sqrt{7})$.

Consider $(120, 43+w)(700, 91+w)$ in the field $k(\sqrt{[-111]})$, $m \equiv 1(4)$, $w = \frac{1+\sqrt{[-111]}}{2}$, $d=-111$. Here $s=3.7.2^5.5$, $r=5$, and $t=1771$ from the conditions $t \equiv 43 \equiv 91(2)$, $t \equiv 43(3)$, $t \equiv 91(5)$, $t \equiv 91(7)$, $(2t+1)^2 \equiv -111(13440)$, giving the product $5(3380, 1771+w)$.

Canonic Form of the Principal Ideal Corresponding to an Irrational Integer.

The principal ideal corresponding to the integer $\alpha+\beta w$, α and β being rational integers prime to each other, may be put into canonic form by taking s as the norm of $\alpha+\beta w$, *i. e.*,

$$s = \begin{cases} \alpha^2 - \beta^2 m \\ \alpha^2 + \alpha\beta - \beta^2 \dfrac{m-1}{4}. \end{cases}$$

Then, for m not congruent to 1 (4), find a and b such that $a\beta + b\alpha = 1$, and hence $(a+bw)(\alpha+\beta w) = a\alpha + b\beta + w$, and we may take $t \equiv a\alpha + b\beta \ m(s)$. For

the case $m \equiv 1$ (4), find a and b such that $(a+b)\beta+b\alpha=1$, and take $t \equiv a\alpha + b\beta\frac{m-1}{4}(s)$.

But, given an ideal in canonic form, it is not readily determined whether it is or is not a principal ideal; and hence we look for the necessary and sufficient conditions that a given ideal $(s, t+w)$ may be the principal ideal corresponding to some integer $\alpha+\beta w$. Consider the relation

(7.) $ls+nt+nw=(a+bw)(\alpha+\beta w)$, l, n, a, b, α, and β all rational integers.

Evidently $(s, t+w)$ will be a principal ideal if, and only if, we can find α and β such that

(1.) When a and b are arbitrarily assigned, l and n can be found to satisfy (7.); and

(2.) When l and n are arbitrarily assigned, a and b can similarly be found.

For simplicity consider the case m not congruent to 1 (4). Equating rational and irrational parts of (7.) we have

$$n=b\alpha+a\beta$$

$$l=\frac{a(\alpha+\beta t)+b(\beta m-\alpha t)}{s},$$

and if l is to be an integer for all values of a and b, we must have

$$\alpha-\beta t \equiv 0 \ (s) \tag{8.}$$

$$\beta m-\alpha t \equiv 0 \ (s) \tag{9.}$$

Again,
$$a=\frac{-l\alpha s-n(\beta m-\alpha t)}{\alpha^2-\beta^2 m}$$

and
$$b=\frac{l\alpha s+n(\beta-\beta t)}{\alpha^2-\beta^2 m}$$

and if a and b are to be integers for all values of l and n, we must have

$$\alpha s \equiv 0 \ (\alpha^2-\beta^2 m) \tag{10.}$$

$$\beta s \equiv 0 \ (\alpha^2-\beta^2 m) \tag{11.}$$

$$\alpha-\beta t \equiv 0 \ (\alpha^2-\beta^2 m) \tag{12.}$$

$$\beta m-\alpha t \equiv 0 \ (\alpha^2-\beta^2 m) \tag{13.}$$

If (8.) is true, then $\alpha t-\beta t^2 \equiv 0$ (s), and since $t^2 \equiv m$ (s) we must have $\alpha t-$

$\beta m \equiv 0\ (s)$. Hence (8.) includes (9.). Again, if α contained any factor of $\alpha^2 - \beta^2 m$, β would also have to contain it; and since α and β are prime to each other, this is impossible. Hence (10.) and (11.) reduce to $s \equiv 0\ (\alpha^2 - \beta^2 m)$, and this in turn brings (12.) and (13.) under (8.). Hence our six conditions reduce to two, namely,

$$\alpha - \beta t \equiv 0\ (s) \tag{8.}$$

$$\text{and } s \equiv 0\ (\alpha^2 - \beta^2 m) \tag{14.}$$

Moreover, since $\alpha - \beta t \equiv 0\ (s)$, we have $\alpha t - \beta t^2 \equiv 0\ (s)$, and hence $\alpha^2 - \beta^2 m \equiv 0\ (s)$. But this, together with (14.), gives us the equation

$$\pm s = \alpha^2 - \beta^2 m \tag{15.}$$

Similarly for the case $m \equiv 1\ (4)$, we find that the conditions redvce to (8.) and the equation

$$\pm s = \alpha^2 + \alpha\beta - \beta^2 \frac{m-1}{4} \tag{16.}$$

Hence the necessary and sufficient condition that $(s, t+w)$ should be a principal ideal is that it should be possible to find rational integers α and β to satisfy equation (15.) or (16.) and congruence (8.).

If s is a prime number, the equation alone is sufficient.

As a special case, the conditions ars evidently fulfilled if the norm of $t+w$, $t^2 - w^2$, is equal to s; in which case $(s, t+w)$ is the principal ideal corresponding to $t+w$.

Equation (15.), or (16.), gives the necessary condition for a principal ideal that $\left(\frac{\pm s}{m}\right) = 1$, or $\left(\frac{\pm 4s}{m}\right) = 1$, while for any ideal whatever we must have

$$\begin{cases} \left(\frac{m}{s}\right) = 1. \\ \left(\frac{m}{4s}\right) = 1. \end{cases}$$

The necessary and sufficient condition in the form to which we have reduced it is of little practical value as a test if $m > 0$. For $m < 0$ it may be useful.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

350. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Solve the equations: $x+y+z=a_0$,
$x+yu+zv=a_1$,
$x+yu^2+zv^2=a_2$,
$x+yu^3+zv^3=a_3$,
$x+yu^4+zv^4=a_4$.

Solution by A. H. HOLMES, Brunswick, Maine, and the PROPOSER.

$x+y+z=a_0\ldots(1)$; $x+yu+zu=a_1\ldots(2)$; $x+yu^2+zv^2=a_2\ldots(3)$; $x+yu^3+zu^3=a_2\ldots(4)$; and $x+yu^4+zu^4=a_4\ldots(5)$.

Subtracting (1) from (2), (2) from (3), (3) from (4), and (4) from (5), we have,

$$y(u-1)+z(v-1)=a_1-a_0\ldots(6);$$
$$yu(u-1)+zu(v-1)=a_2-a_1\ldots(7);$$
$$yu^2(u-1)+zv^2(v-1)=a_3-a_2\ldots(8);\text{ and}$$
$$yu^3(u-1)+zv^3(v-1)=a_4-a_3\ldots(9).$$

Eliminating y from (6), (7), (8), and (9), we have,

$$(a_1-a_0)u-zu(v-1)=a_2-a_1-zv(v-1)\ldots(10),$$
$$(a_2-a_1)u-zuv(v-1)=a_3-a_2-zv^2(v-1)\ldots(11),\text{ and}$$
$$(a_3-a_2)u-zuv^2(v-1)=a_4-a_3-zv^3(v-1)\ldots(12).$$

Eliminating z from (10), (11), and (12), we have,

$$(a_1-a_0)uv-(a_2-a_1)v=(a_2-a_1)u-(a_3-a_2)\ldots(13),\text{ and}$$
$$(a_2-a_1)uv-(a_3-a_2)v=(a_3-a_2)u-(a_4-a_3)\ldots(14).$$

Eliminating v from (13) and (14), we have

$$\frac{(a_3-a_2)u-(a_4-a_3)}{(a_2-a_1)u-(a_3-a_2)}=\frac{(a_2-a_1)u-(a_3-a_2)}{(a_1-a_0)u-(a_2-a_1)}.$$

Putting $a_1-a_0=d_1$, $a_2-a_1=d_2$, $a_3-a_2=d_3$, $a_4-a_3=d_4$, and solving,

$$u=\frac{d_1d_4-d_2d_3\pm\sqrt{[(d_1d_4-d_2d_3)^2+4(d_1d_3-d_2^2)(d_2^2-d_2d_4)]}}{2(d_1d_3-d_2^2)}.$$

From (13) we find v. Then from (10) we find z. From (6), y, and from (1), x.

351. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve, $y^2+yz+z^2=a^2$... (1).
$z^2+zx+x^2=b^2$... (2).
$x^2+xy+y^2=c^2$... (3).

I. Solution by J. A. COLSON, Searsport, Maine.

$b^2-c^2=(z-y)(x+y+z)$. $\therefore (b^2-c^2)x=(zx-xy)(x+y+z)$.

$c^2-a^2=(x-z)(x+y+z)$. $\therefore (c^2-a^2)y=(xy-yz)(x+y+z)$.

$a^2-b^2=(y-x)(x+y+z)$. $\therefore (a^2-b^2)z=(yz-zx)(x+y+z)$.

$\therefore (b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z=0$.

For convenience, put $b^2-c^2=f$, $c^2-a^2=g$, and $a^2-b^2=h$. Then $f+g+h=0$, and $fx+gy+hz=0$.

$$\therefore z=-\frac{fx+gy}{h}, \text{ and } x+y+z=x+y-\frac{fx+gy}{h}=\frac{(h-f)x+(h-g)y}{h}.$$

$$\therefore a^2-b^2=h=(y-x)(x+y+z)=(y-x)\frac{(h-f)x+(h-g)y}{h}.$$

$\therefore h^2=(f-h)x^2+(g-f)xy+(h-g)y^2$.

But from (3) we have $y^2=c^2-x^2-xy$.

Hence, $h^2=c^2(h-g)+(f+g-2h)x^2+(2g-f-h)xy=c^2(h-g)-3hx^2+3gxy$.

$$\therefore y=\frac{3hx^2+h^2+c^2(g-h)}{3gx}.$$

Substitute in (3), and we have

$$x^2+\frac{3hx^2+h^2+c^2(g-h)}{3g}+\frac{[3hx^2+h^2+c^2(g-h)]^2}{9g^2x^2}-c^2=0.$$

Hence, clearing of fractions and uniting, we have,

$$9(g^2+gh+h^2)x^4-3[c^2(2g^2-gh+2h^2)-h^2(g+2h)]x^2+[h^2+c^2(g-h)]^2=0.$$

$$\therefore 36(g^2+gh+h^2)^2x^4-12(g^2+gh+h^2)[c^2(2g^2-gh+2h^2)-h^2(g+2h)]x^2+[c^2(2g^2-gh+2h^2)-h^2(g+2h)]^2=[c^2(2g^2-gh+2h^2)-h^2(g+2h)]^2-4(g^2+gh+h^2)[h^2-c^2(g-h)]^2=9c^4g^2h^2-6c^2g^2h^2(2g+h)-3g^2h^4.$$

$$\therefore 6(g^2+gh+h^2)x^2-[c^2(2g^2-gh+2h^2)-h^2(g+2h)]=\pm ghi\sqrt{[9c^4-6c^2(2g+h)-3h^2]}.$$

Giving g and h their original values, we have

$$6(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)x^2=2b^6+2c^6-a^6+4a^4(b^2+c^2)$$
$$-5a^2(b^4+c^4)-5a^2(b^4+c^4)+b^2c^2(b^2+c^2-3a^2)$$
$$\pm(a^2-b^2)(c^2-a^2)\sqrt{[3(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)]}.$$

If k=the area of a triangle whose sides are a, b, and c, then $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4=16k^2$.

Hence, $6(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)x^2$
$$=2b^6+2c^6-a^6+4a^4(b^2+c^2)-5a^2(b^4+c^4)$$
$$b^2c^2(b^2+c^2-3a^2)\pm4(a^2-b^2)(c^2-a^2)k\sqrt{3}.$$

Hence, by permuting the letters a, b, c we can find the values of y^2 and z^2 from the two following equations:

$$6(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)y^2=2c^6+2c^6+2a^6-b^6+4b^4(c^2+a^2)$$
$$-5b^2(c^4+a^4)+c^2a^2(c^2+a^2-3b^2)\pm4(b^2-c^2)(a^2-b^2)(k\sqrt{3}$$
and $$6(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)z^2=2a^6+2b^6-c^6+4c^4(a^2+b^2)$$
$$-5c^2(a^4+b^4)+a^2b^2(a^2+b^2-3c^2)\pm4(c^2-a^2)(b^2-c^2)k\sqrt{3}.$$

II. Solution by ARTEMAS MARTIN, LL. D., Editor and Publisher, Mathematical Magazine, Washington, D. C.

From the square of the sum of the given equations, subtract twice the sum of their squares and extract the square root of one-third of the remainder; then

$$xy+yz+xz=\pm\tfrac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\ldots(4).$$

Subtracting twice (1) from the sum of (4) added to the sum of the given equations,

$$2x(x+y+z)=b^2+c^2-a^2\pm\tfrac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\ldots(5).$$

Subtracting twice (2) and twice (3) in succession from the same sum,

$$2z(x+y+z)=a^2+c^2-b^2\pm\tfrac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\ldots(6),$$
$$2z(x+y+c)=a^2+b^2-c^2\pm\tfrac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\ldots(7).$$

Add the three equations (5), (6), and (7); then

$$2(x+y+z)(x+y+z)=2(x+y+z)^2=a^2+b^2+c^2$$
$$\pm\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\ldots(8).$$

Extracting the square root of twice (8),

$$2(x+y+z)=\pm\sqrt{\{2(a^2+b^2+c^2)\pm 2\sqrt{[3(a^2+b^2+c^2)^2-6(a^4+b^4+c^4)]}\}} \quad ..(9).$$

Dividing (5), (6), and (7) in succession by (9),

$$x=\frac{b^2+c^2-a^2\pm\frac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm 2\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}\}}},$$

$$y=\frac{a^2+c^2-b^2\pm\frac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm 2\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}\}}},$$

$$z=\frac{a^2+b^2-c^2\pm\frac{1}{3}\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm 2\sqrt{[3(a^2+b^2+c^2)^2-(a^4+b^4+c^4)]}\}}}.$$

Also solved by A. H. Holmes, J. Scheffer, and V. M. Spunar.

For a number of different solutions of this problem, when the known quantities are not squared, see *The Mathematical Magazine*, published by Dr. Artemas Martin, Vol. II, pp. 141-144, and pp. 193-196. ED. F.

GEOMETRY.

375. Proposed by C. N. SCHMALL, New York City.

From a point P on a circle there are drawn three chords PA, PB, PC. Show that the circles described on these chords as diameters intersect again in three collinear points.

I. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology.

Take the point P as the origin of polar coördinates, and the diameter through P as the initial line. The coördinates of the points A, B, and C are, respectively, $2a\cos\alpha$, α; $2a\cos\beta$, β; $2a\cos\gamma$, γ; α, β, and γ being the vectorial angles.

The equations of the circles described upon the chords as diameters will be

$$\rho=2a\cos\alpha\cos(\theta-\alpha),$$
$$\rho=2a\cos\beta\cos(\theta-\beta),$$
$$\rho=2a\cos\gamma\cos(\theta-\gamma),$$

whence the coördinates of the points of intersection are

$$2a\cos\beta\cos\gamma,\ \beta+\gamma;\quad 2a\cos\gamma\cos\alpha,\ \gamma+\alpha;\quad 2a\cos\alpha\cos\beta,\ \alpha+\beta.$$

These points are all on the straight line whose equation is

$$2a\cos\alpha\cos\beta\cos\gamma=\rho\cos(\theta-\alpha-\beta-\gamma).$$

Join the points A, B, and C to make a triangle. The points of intersection are then the feet of the perpendiculars let fall from P upon the three sides, and the line through the points of intersection is the pedal line of P with respect to the triangle.

II. Solution by S. LEFSEHETZ, Clark University.

If we transform by inversion, the pole of inversion being in P, the transformed of the three circles of diameters PA, PB, and PC are perpendiculars at PA, PB, and PC in A', B', and C', points where these three lines meet the line obtained by transformation of the given circle. These three perpendiculars envelop a parabola of focus P; therefore, the circle circumscribed to the triangle they form passes through P, — a well known property of the parabola. By transforming back, we obtain a straight line and the proposition is thus proved.

Also solved by the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

355. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Solve the equations:

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=a_1;$$
$$\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=a_2;$$
$$\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}=a_3;$$
$$\cdots\cdots\cdots$$
$$\frac{1}{x_{n-2}}+\frac{1}{x_{n-1}}+\frac{1}{x_n}=a_{n-2};$$
$$\frac{1}{x_{n-1}}+\frac{1}{x_n}+\frac{1}{x_1}=a_{n-1};$$
$$\frac{1}{x_n}+\frac{1}{x_1}+\frac{1}{x_2}=a_n.$$

356. Proposed by ARTEMAS MARTIN, Ph. D., Washington, D. C.

Solve by quadratics, if possible, the equations,

$$w(x+y+z)=a, \quad x(w+y+z)=b,$$
$$y(w+x+z)=c, \quad z(w+x+y)=d.$$

[From the *Mathematical Magazine*, Vol. II, p. 256.]

GEOMETRY.

387. Proposed by DANIEL KRETH, Oxford, Iowa.

A lot 100 feet long and 60 feet wide, has a walk extending from one corner half way around it, and occupying one-third of the area. Required the width of the walk. A geometrical construction is desired.

388. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

A conic is inscribed in a triangle and one focus lies on the polar circle of the triangle. Prove that the corresponding directrix passes through the center of perpendiculars.

389. Proposed by H. PRIME, Boston, Mass.

On the same side of a given base, triangles are erected such that the bisectors of their vertex angles all pass through a given point. Find the locus of the vertices (i) when the vertex angle are all equal, (ii) when the vertex angles are all unequal.

CALCULUS.

311. Proposed by WILMER THOMPSON, Senior, Drury College.

Solve the differential equation,

$$\left(\frac{dy}{dx}\right)^3+x^3=ax\left(\frac{dy}{dx}\right).$$

[From Forsyth's *Differential Equations*, p. 47.]

312. Proposed by C. N. SCHMALL, New York City.

Given $y^3-3y+x=0$, prove by Maclaurin's theorem, that

$$y=\frac{x}{3}+\frac{x^3}{3^4}+\frac{x^5}{3^6}+\text{etc.}$$

MECHANICS.

261. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

A man six feet high, walking at a rate of 100 yards a minute, crosses a muddy road close behind a wheel of a carriage which is going thrice as fast and in a direction at right angles to that of the man's motion. The diameter of the wheel is five feet. If, when the man is four feet from the middle of the wheel the mud is splashed up to the hight of seven feet, will any of it touch him? Unsolved in *Educational Times*.

262. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

A hemispherical shell, whose radius is equal to the mean radius of the earth and whose thickness is one centimeter, is constructed of a matter whose density is equal to the mean density of the earth. A particle starts from rest at the center of the shell under the action of the attraction of the shell. Express as the decimal of a year the time it takes the particle to reach the surface of the shell, and find the velocity in centimeters per second of the particle just before it reaches the shell. Unsolved in *Educational Times*.

NOTES AND NEWS.

Problems for solution in the several departments from various contributors are desired. Please send us good problems for solution and do not let a few do all the proposing of problems. F.

We learn, from *Science*, that Professor Arthur Ray Maxson, instructor in mathematics at Columbia University, died April 13, at the age of thirty years. Professor Maxson was for several years a valuable contributor to the MONTHLY, and we shall miss his loyalty and support. F.

We are receiving numerous complaints from subscribers who fail to get their numbers of the MONTHLY. Our only answer is that we mail a copy to each of our subscribers, and that the failure to receive them must lie in the great majority of cases with the Postal Department. We hope our subscribers will bear patiently with us in this matter until the Department gets through punishing the big magazines and political delinquents for their unfriendly criticisms, after which it is hoped some attention will be given to efficiency and public service. The MONTHLY is mailed on the 28th of each month. F.

The last number of the *L'Enseignement Mathématique*, March, 1911, page 148, contains some details in regard to the *Encyclopedia of elementary mathematics* which the Italian mathematical society "Mathesis" has had under consideration for more than a year. The work will contain forty-four monographs under the following headings: Logic of mathematics; elementary arithmetic; theory of numbers; the notion of number and its extensions; limits, series, continued fractions, and infinite products; progressions and logarithms; literal calculus and algebraic identities; combinatory analysis, determinants, and linear equations; equations whose degree exceeds one; algebraic problems and their discussion; elements of the infinitesimal calculus; relations between analysis and elementary algebra; elementary properties of plane and space figures; theory of measure and applications; geometry of the triangle and of the tetrahedron; regular polygons and regular polyhedrons; elementary geometric transformations; linear systems of circles and spheres; geometry on the sphere; sections of the cylinder and the circular cone; maxima and minima in geometry; methods of solution of geometric problems and classic problems; foundations of elementary geometry; circular and hyperbolic functions, and plane and spherical trigonometry; vectorial calculus; elements of analytic geometry; elements of projective geometry; elements of descriptive geometry; special curves and surfaces; non-euclidean geometry; non-archimedean geometry; geometric representation of complex numbers; relations between elementary geometry and the theories of higher geometry; units of measure; numerical approximations and graphic calculus; calculus of probability and the theory of errors;

elementary applications of mathematics to the physical sciences; mathematics of statistics; mathematics of finance; history of elementary mathematics; didactic methods and text-books; mathematical recreations; instruments; models.

From these headings, and from the standing of the men who have undertaken the preparation of these monographs, it appears that Italy will soon have a work which will be extremely useful to her teachers of secondary mathematics. It is to be hoped that its appearance will tend to attract attention to the great need of an extensive mathematical encyclopedia in English, dealing with secondary mathematics. Those whose interests are mainly in higher mathematics are generally in position to use the great encyclopedia which is now being published in French and German, but teachers of secondary mathematics cannot usually derive so much profit from this great work. It would appear that our need of an encyclopedia of elementary mathematics should be greater than the need in Italy, and that this need should receive attention at our teachers meetings, especially at those having a national scope. M.

BOOKS.

Introduction A La Théorie de Nombres Algébriques. Par Dr. J. Sommer, Professeur A La "Technische Hochschule" de Dantzig. Edition Francaise Revue et Augmenteé. Traduit de L'Allemend Par A. Lévy, Professeur Au Lycée Saint-Louis. Avec Préface de J. Hadamard, Professeur Au Collége de France. Paper Cover, x+376 pages. Price, 15 francs. Paris: A. Hermann et Fils.

This work is divided into five chapters. The first chapter is the Introduction, treating the theorems of ordinary number theory. Chapter II deals with quadratic bodies; Chapter III, applications of the theory of quadratic bodies; Chapter IV, bodies of the third degree; and Chapter V, relative bodies. The treatment of the various topics indicated in the above enumeration is very clear and free from unnecessary complex and abstruce statements. In Chapter III, pages 184-201, we find a discussion of the last theorem of Fermat. We especially recommend a careful reading of this discussion. It will clear up a number of difficulties connected with this famous problem, interest in which has recently been revived by an offer of 100,000 marks for a proof or disproof of its truth. F.

Théorie des Corps de Nombres Algébriques. Par David Hilbert, Professeur A L'Université de Goettingen. Ouvrage traduit de L'Allemand Par A. Lévy, Professeur Aut Lycée Saint-Louis. Première Partie Le Nombre Algébriques et la Corps Algébriques. Paper Cover, 72 pages. Price, 3 francs. Paris: A. Hermann et Fils.

This translation places the profound researches of Hilbert on the theory of ideals in French garb, and enables English readers to read them both in the original and in the French. F.

A Text-book of Differential Calculus, with Numerous Worked Out Examples. By Ganesh Prasad, B. A. (Cantab), D. Sc. (Allahabad). 8vo. Cloth, xii+161 pages. Price, $1.50. New York and London: Longmans, Green & Co.

This work aims to establish the fundamental principles of the Differential Calculus on a firm foundation and to present its first principles so clearly as to be readily understood by the beginner. The book closes with a few miscellaneous notes among which is a discussion of Weierstrass's Function. F.

The Technical World Magazine for March contains among other articles of interest, one on the Story of the Speaking Dog. The actual breaking into human speech by this animal is one of the most extraordinary things of modern times. F.

The American Journal of Mathematics for April contains the following articles: On Three-Spreads Satisfying Four or More Homogeneous Linear Partial Differential Equations of the Second Order, by Charles H. Sisam; Some Properties of Lines in Space of Four Dimensions and Their Interpretation in the Geometry of the Circle in Space of Three Dimensions, by C. L. E. Moore; On the Geometry of Line Elements in the Plane with Reference to Oscillating Circles, by George F. Gundelfinger; Binary Modular Groups and Their Invariants, by Leonard E. Dickson; The Group of Turns and Slides and the Geometry of Turbines, by Edward Kasner. F.

The Transactions of the American Mathematical Society for April contains the following articles: Biorthogonal Systems of Functions, by Anna J. Pell; Applications of Biorthogonal Systems of Functions to the Theory of Integral Equations, by Anna J. Pell; On the Uniform Convergence of the Developments in Bessel Functions, by C. N. Moore; Determination of the Ordinary and Modular Ternary Linear Groups, by H. H. Mitchell; General Theory of Linear Differential Equations, by Geo. D. Birkhoff. F.

The Mathematical Magazine. A Journal Devoted to Elementary Mathematics. Edited and Published by Artemas Martin, Ph. D., LL. D., Washington, D. C. Issued Quarterly. Price, $1.00 per year, in advance.

The last number, viz., No. 12, of Vol. II, recently published, contains the following articles: About Sixth Power Numbers whose Sum is a Sixth Power, by Artemas Martin; Elementary Proof of Properties of Numbers, by Rollin A. Harris; The United States Sinking-Fund, by Theodore L. DeLand; On Rational Scalene Triangles, by Artemas Martin; On Biquadrate Numbers, by Benton Haldeman; On Rational Right-Angled Triangles, by Artemas Martin.

It is to be regretted that Dr. Martin does not have the leisure to publish the magazine regularly. *The Mathematical Magazine* and *The Mathematical Visitor*, both edited and published by Dr. Martin, even the type-setting being done by himself, are the finest specimens of the type-setters' art in mathematical typography. F.

The Problem of the Angle-Bisectors. A Dissertation Submitted to the Ogden Graduate School of Science of the University of Chicago in Candidacy for the Degree of Doctor of Philosophy, by Richard Philip Baker. 98 pages. Price, $1.10. The University of Chicago Press.

This Thesis is a very detailed discussion of the famous problem, "To construct a triangle, having given the three angle bisectors." This problem, which might easily be attacked by a freshman with the hope of success, proves to be a very intractable one, and not amenable to the ordinary processes of Euclidean Geometry. Dr. Baker has performed a very creditable piece of work in the production of this interesting thesis. F.

5

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SHOULD READ ISSUE 5

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NUMBER.

By LOUIS C. KARPINSKI, University of Michigan.

Some fundamental errors in regard to the nature and origin of number have been given such wide circulation in recent works on psychology and the psychology of number that it seems desirable to present views which are based upon the scientific and revolutionary work upon the nature of the number idea which has been done by Cantor,* Dedekind, Peano, Frege, and others. Psychologists, philosophers, and pedagogs have treated the subject of the definition of number and the genesis of the number idea in apparent ignorance of the fact that great mathematicians have also labored in this field. While some may dispute the right of mathematicians to discuss so psychological a matter as the genesis of the number idea, no one can dispute that a somewhat logical definition of number is a necessary basis for work on the genesis of the number idea.

One faulty definition which persists is that number is ratio. If that be so, pray what is ratio? Surely the idea of ratio is no more evident than is that of number.

Upon this poor foundation a method of teaching arithmetic is based whose merit, if any, is in spite of this fundamentally worthless concept of number. To confuse number with ratio is to confuse number with an application of number. Logically ratio is an unsymmetric relation between two numbers.

Some writers would have us believe that number takes its origin in counting. Here, again, we may ask, is counting intuitive? Is the idea of counting any simpler than the idea of number? Counting presupposes the presence of the number idea in the mind of the counter. Children often go through the counting process in much the same way that they repeat Old Mother Goose jingles; as much number concept is present in one operation as in the other. The same criticism holds against that view of number

* Cantor, Georg, *Mathematische Annalen*, Vol. XLVI, p. 481, *Beitrage zur Begründung transfiniten Mengenlehre*. Dedekind, R. *Was sind und was sollen die Zahlen?* Also translated into English by W. W. Beman in *Essays on the Theory of Numbers*, Open Court Co. Frege, G. *Grundlagen der Arithmetic*.